

CHAPTER 10

DEFORMATION

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Consider the bar in figure 1 subjected to a simple tension loading F .

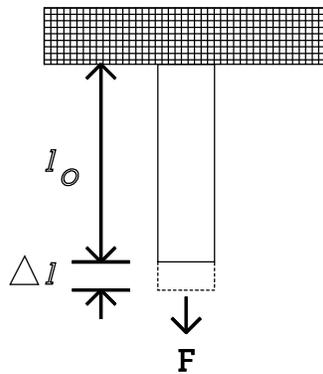


Figure 1: Bar in Tension

Engineering Stress (σ) is the quotient of load (F) and area (A). The units of stress are normally pounds per square inch (psi).

$$\sigma = \frac{F}{A}$$

where:

σ	is the stress (psi)
F	is the force that is loading the object (lb)
A	is the cross sectional area of the object (in^2)

When stress is applied to a material, the material will deform. Elongation is defined as the difference between loaded and unloaded length

$$\Delta l = L - L_0$$

where:

Δl	is the elongation (ft)
L	is the loaded length of the cable (ft)
L_0	is the unloaded (original) length of the cable (ft)

Strain is the concept used to compare the elongation of a material to its original, undeformed length. Strain (ϵ) is the quotient of elongation (e) and original length (L_0). Engineering Strain has no units but is often given the units of in/in or ft/ft.

$$\epsilon = \frac{\Delta l}{L_0}$$

where: ϵ is the strain in the cable (ft/ft)
 Δl is the elongation (ft)
 L_0 is the unloaded (original) length of the cable (ft)

Example Find the strain in a 75 foot cable experiencing an elongation of one inch.

Solution:

$$\text{Strain}(\epsilon) = \frac{e(\text{ft})}{L_0(\text{ft})} = \frac{1 \text{ in } (1\text{ft}/12\text{in})}{75 \text{ ft}} = 1.11 \times 10^{-3} \text{ ft / ft}$$

10.1 Stress-Strain Diagrams and Material Behavior

Stress and strain are calculated from easily measurable quantities (normal load, diameter, elongation, original length) and can be plotted against one another as in Figure 2. Such Stress-Strain diagrams are used to study the behavior of a material from the point it is loaded until it breaks. Each material produces a different stress-strain diagram.

Point 1 on the diagram represents the original undeformed, unloaded condition of the material. As the material is loaded, both stress and strain increase, and the plot proceeds from Point 1 to Point 2. If the material is unloaded before Point 2 is reached, then the plot would proceed back down the same line to Point 1.

If the material is unloaded anywhere between Points 1 and 2, then it will return to its original shape, like a rubber band. This type of behavior is termed *Elastic* and the region between Points 1 and 2 is the *Elastic Region*.

The Stress-Strain curve also appears linear between Points 1 and 2. In this region stress and strain are proportional. The constant of proportionality is called the *Elastic Modulus* or *Young's Modulus* (E). The relationship between stress and strain in this region is given by Equation 5-4.

$$E = \frac{\sigma}{\epsilon} \quad \text{or} \quad \sigma = E\epsilon$$

where: σ is the stress (psi)
 E is the Elastic Modulus (psi)
 ϵ is the strain (in/in)

The Elastic Modulus is also the slope of the curve in this region, solved by taking the slope between data points (0,0) and (σ_y, ϵ_y) .

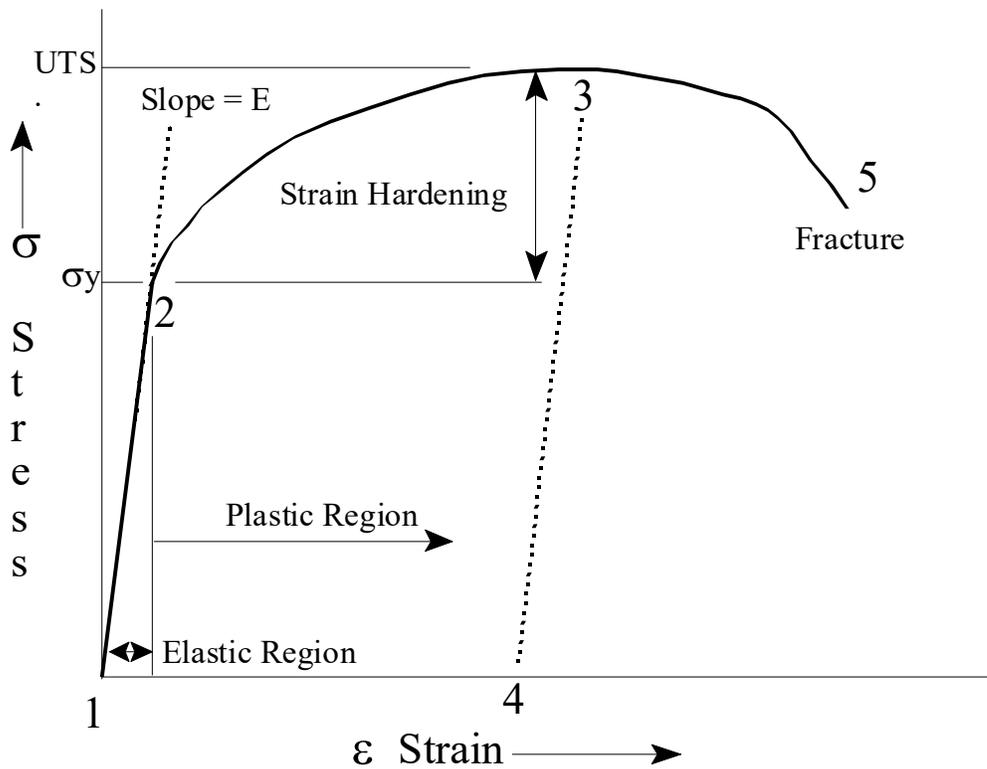


Figure 2 Stress/Strain Diagram

Point 2 is called the *Yield Strength* (σ_y). If it is passed, the material will no longer return to its original length. It will have some permanent deformation. This area beyond Point 2 is the *Plastic Region*. Consider, for example, what happens if we continue along the curve from Point 2 to Point 3, the stress required to continue deformation increases with increasing strain. If the material is unloaded the curve will proceed from Point 3 to Point 4. The slope (Elastic Modulus) will be the same as the slope between Points 1 and 2. The difference between Points 1 and 4 represents the permanent strain of the material.

If the material is loaded again, the curve will proceed from Point 4 to Point 3 with the same Elastic Modulus (slope). The Elastic Modulus will be unchanged, but the Yield Strength will be increased. Permanently straining the material in order to increase the Yield Strength is called *Strain Hardening*.

If the material is strained beyond Point 3 stress decreases as non-uniform deformation and necking occur. The sample will eventually reach Point 5 at which it fractures.

The largest value of stress on the diagram is called the *Tensile Strength (TS)* or *Ultimate Tensile Strength (UTS)*. This is the most stress the material can support without breaking.

Example: (This is from the FE Exam) A Steel rod ($E=200$ GPa) has a circular cross section and is 10m long. Determine the minimum diameter if the rod must hold a 30 kN tensile force without deforming more than 5mm. Assume the steel stays in the elastic region. Note, 1 GPa = 10^9 Pa.

Solution:

Knowing the initial length and the change in length permits the calculation of strain.

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{5\text{mm}(\frac{1\text{m}}{1000\text{mm}})}{20\text{m}} = 0.0005$$

In the elastic region, the stress σ is directly proportional to the strain ε , by the Modulus of Elasticity, E

$$\frac{F}{A_0} = \sigma = E\varepsilon$$

Rearranging, substituting values and converting units,

$$\sigma = E\varepsilon = (200 \text{ GPa})0.0005 = 0.1 \text{ GPa} = 0.1 \times 10^9 \text{ Pa} = 0.1 \times 10^9 \text{ N/m}^2$$

The definition of stress $\sigma = \frac{F}{A_0}$ can be used to find the required cross section area.

$$A_0 = \frac{F}{\sigma} = \frac{30\text{kN}(\frac{1000\text{N}}{\text{kN}})}{0.1 \times 10^9 \text{ N/m}^2} = 0.0003\text{m}^2(\frac{1000\text{mm}}{\text{m}})(\frac{1000\text{mm}}{\text{m}}) = 300\text{mm}^2$$

The diameter, d_0 is solved from the area of a circle

$$A_0 = \frac{\pi d_0^2}{4}$$

$$d_0^2 = \frac{A_0 4}{\pi}$$

$$d_0 = \sqrt{\frac{A_0 4}{\pi}} = \sqrt{\frac{300\text{mm}^2 * 4}{3.14}} = 19.5\text{mm}$$

10.2 Material Characteristics

There are five material properties that do a good job at describing the characteristics of a material. They are strength, hardness, brittleness, toughness, and ductility. Each of these will be discussed in more detail in the upcoming chapters.

Strength is a measure of the material's ability to resist deformation and to maintain its shape. Strength can be quantified in terms of yield stress or ultimate tensile strength. Both yield stress and ultimate tensile strength can be determined from tensile test data by plotting a stress-strain curve.

Hardness is a measure of the material's ability to resist indentation, abrasion, and wear. Hardness is quantified by arbitrary hardness scales such as the Rockwell hardness scale or the Brinell hardness scale. These measurements are obtained by a special apparatus that uses an indenter that is loaded with standard weights. The indenter can have various shapes such as a pyramid or a sphere and is pressed into the specimen. Either the depth of penetration or the diameter of the indentation made is measured to quantify material hardness. Hardness and strength correlate well because both properties are related to inter-molecular bonding. Figure 3 illustrates the test and gives conversion scales for relating values to tensile strength.

Ductility is a measure of a material's ability to deform before failure. Ductility can be quantified by reading the value of strain at the fracture point on the stress-strain curve or by doing a percent reduction in area calculation. Low carbon steels, pure aluminum, copper, and brass are examples of ductile materials.

Brittleness is a measure of a material's inability to deform before failure. Brittleness is the opposite of ductility. Brittleness is not quantified since it is the inability to deform. However, ductility is quantified as discussed above. Examples of brittle materials include glass, cast iron, high carbon steels, and many ceramic materials. Figure 4 shows the difference between ductile and brittle behavior on a stress-strain diagram. This will be discussed in more detail in the chapter on fracture.

Toughness is a measure of a material's ability to absorb energy. There are two measures of toughness: Material Toughness and Impact Toughness. Material Toughness can be measured by calculating the area under the stress-strain curve from a tensile test (Fig 5). The units on this measure of toughness are in-lb/in³. These are units of energy per volume. *Material Toughness* equates to a slow absorption of energy by the material. Impact Toughness is measured by doing a Charpy V-notch Test, discussed further in the chapter on Fracture.

The material toughness can be expressed as the work W required to strain a bar to engineering strain ϵ .

$$W = \int_{l_0}^l F dl = A_0 l_0 \int_0^{\epsilon} \sigma d\epsilon$$

or if w denotes the work per unit volume of material $W/A_0 l_0$, then

$$w = \int_0^{\epsilon} \sigma d\epsilon$$

Thus, the area under the engineering stress-strain curve is a direct measure of the amount of work per unit volume of the material needed to effect a given engineering strain ϵ .

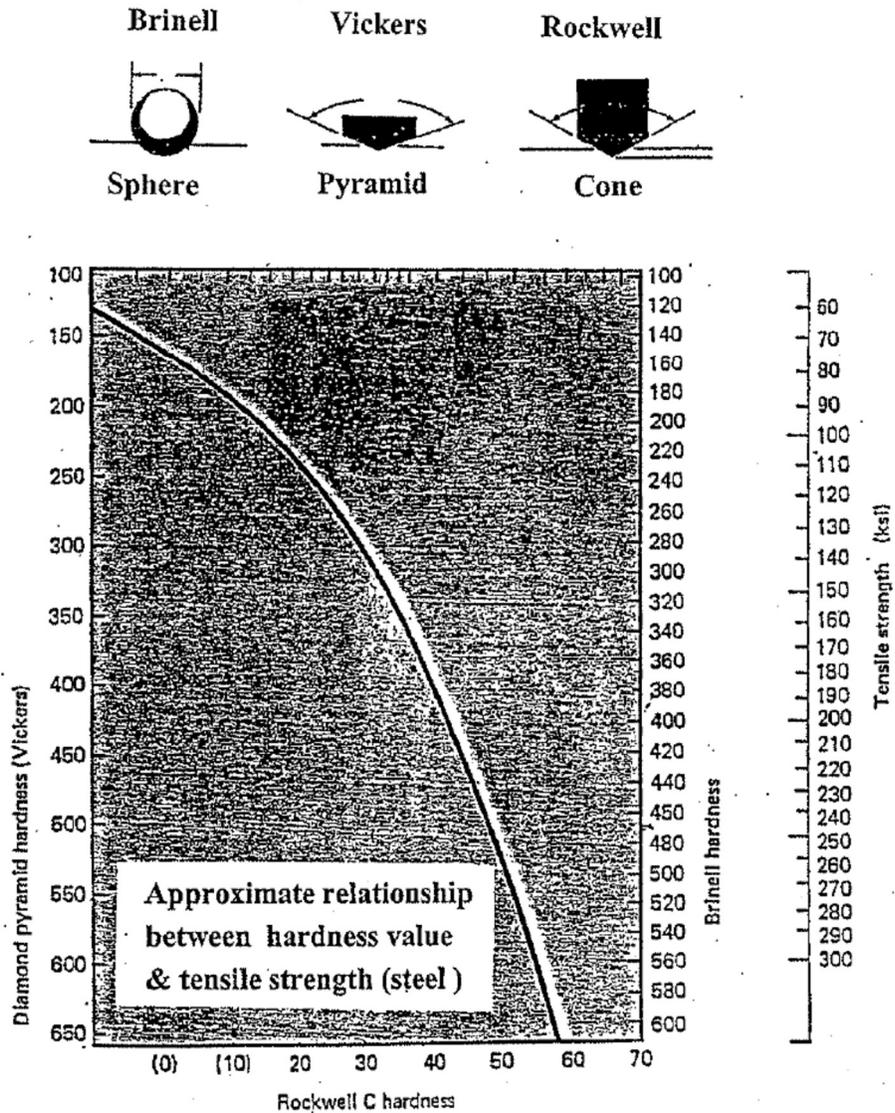


Figure 3: Hardness Tests

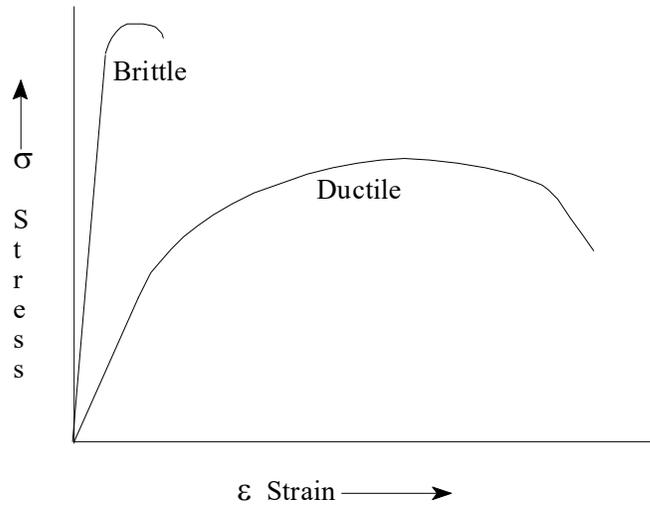


Figure 4. Ductile and Brittle Behavior

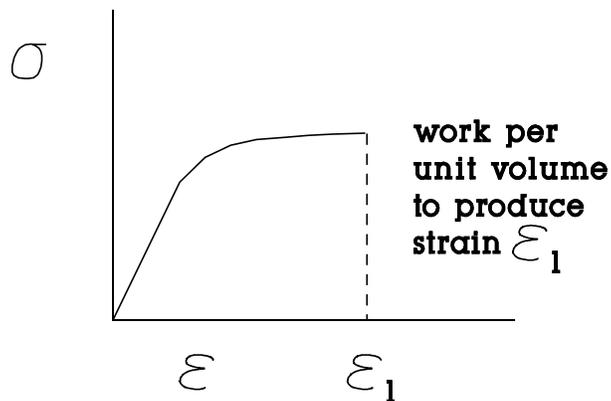


Fig. 5

10.3 Elastic-Plastic Response of Metals

When an ordinary piece of metal is subjected to a force of gradually increasing intensity, the atoms within the individual grains are simply displaced from their initial positions in an elastic fashion; if the force is reduced, the atoms will return to their initial positions.

When the force becomes sufficiently large, the atoms essentially shift from their elastically distorted initial positions to new, identical positions within the lattice, resulting in permanent plastic strain. Thus, in the ordinary tensile test of a metal, we may think of the elastic region as the region where the atoms are slightly displaced from initial positions, whereas the plastic region, planes of atoms are actually slipping over one another, resulting in permanent, plastic deformation.

The actual mechanism by which the atoms slip over one another is generally thought to involve the idea of dislocations or imperfections in the otherwise perfect crystalline arrangement of the atoms. These imperfections (Fig 6) make the slipping (Fig 7) relatively easy.

The two general types of locations are edge dislocations and screw dislocations (Fig. 8). Edge dislocations exist when an extra plane of atoms exists in a portion of the lattice structure. Screw dislocations result when a plane of atoms shifts its orientation from one plane to another.

Force analyses show that slipping of the plates can occur with much lower stress than would be required if the arrangement of atoms was perfect. In fact, theoretical calculations accounting for a given dislocation density are consistent with those found experimentally.

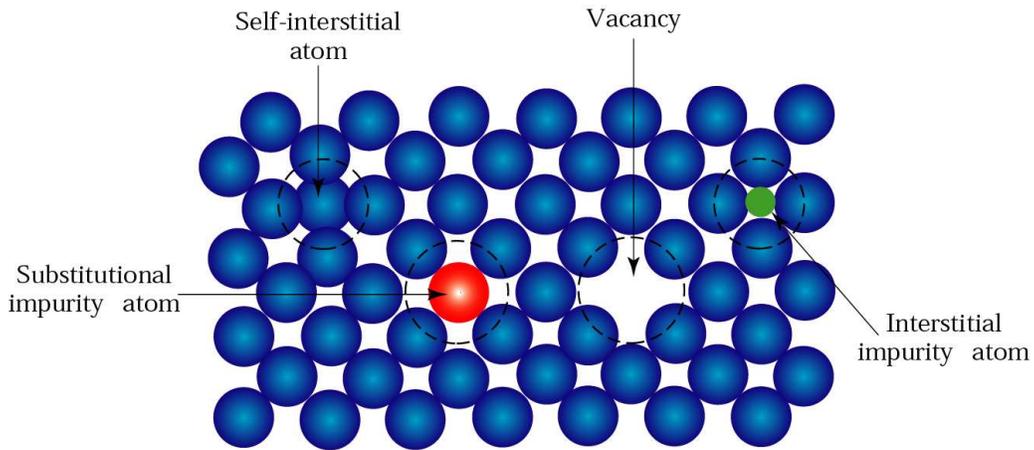


Fig. 6 Interstitial Atoms and Vacancies

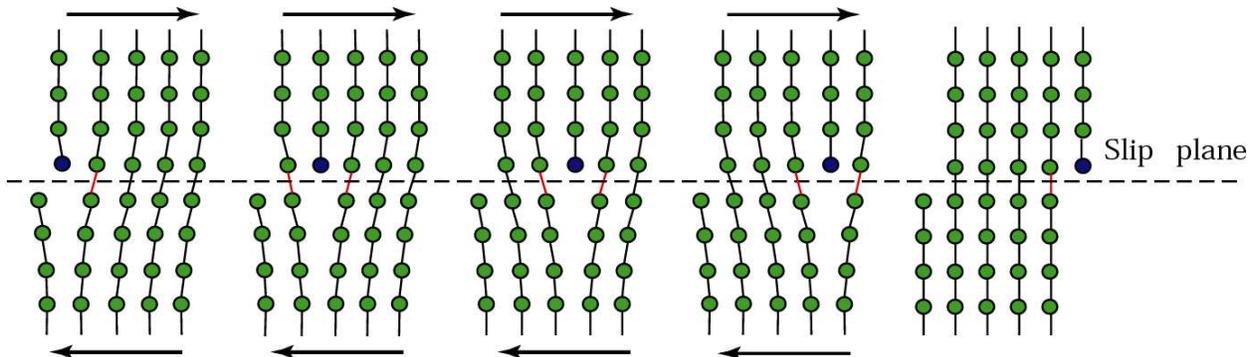


Fig.7 Movement of Edge Dislocations

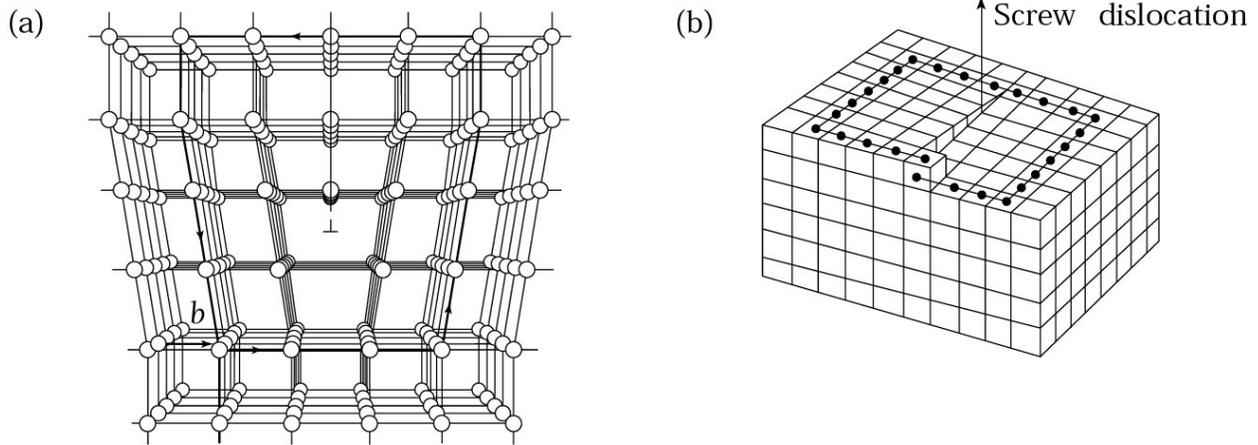


Fig. 8 Edge Dislocation (a) and Screw Dislocation (b)

10.4 True stress and strain measures.

Instead of using engineering stress and strain measures, we may use the so-called "true" stress, σ_T , and "true" strain, ϵ_T , measures. These are defined as

$$\sigma_T = \frac{F}{A}$$

$$\epsilon_T = \int_{l_0}^l \frac{dl}{l} = \ln \left(\frac{l}{l_0} \right)$$

where A denotes the deformed area of the specimen at load F and l denotes the deformed length of the specimen associated with a small increase in extension dl .

In the elastic region, the difference between A and A_0 and l and l_0 is very small so that the "true" stress and strain values are essentially the same as the engineering stress and strain values. When large plastic deformation takes place, the difference is no longer negligible and it is necessary to specify which stress and strain measures are used.

For plastic deformations where the difference between A and l and A_0 and l_0 are important, it has been shown experimentally that the volume of the specimen remains essentially constant such that

$$Al = A_0l_0$$

We also have by the definition of engineering strain

$$l = l_0(1 + \epsilon)$$

Combining the above equations gives us "true" stress and strain expressible as

$$\sigma_T = \sigma(1 + \epsilon), \quad \epsilon_T = \ln(1 + \epsilon)$$

These equations are useful for determining "true" stress-strain response. Notice, however, that since the "true" stress and strain are expressible in terms of the engineering stress and strain, in the real sense one is no more true than the other. One set can be obtained from the other simply by computation. Figure 9 shows a comparison between engineering and "true" stress-strain response.

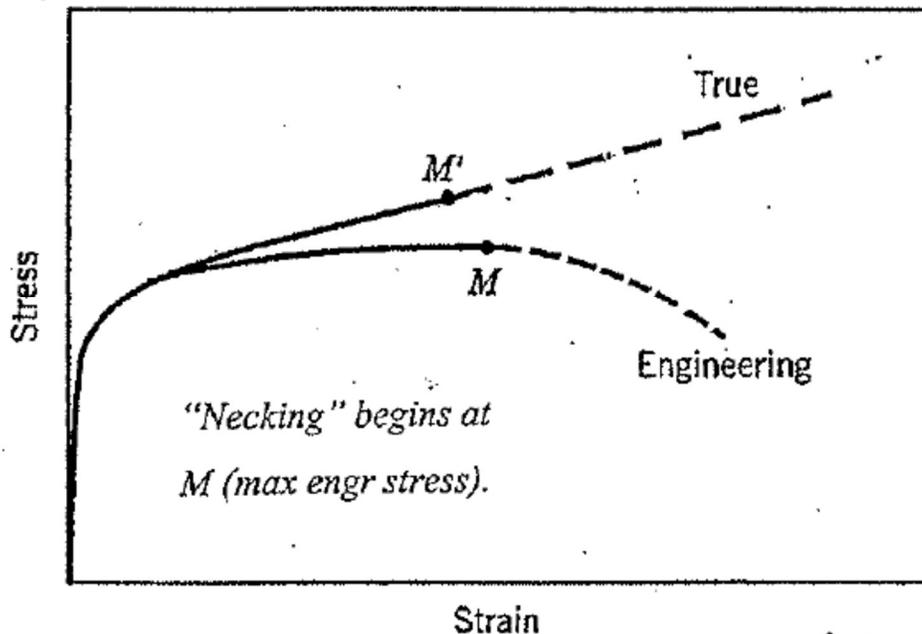


Figure 9: True vs Engineering Stress

10.5 Yielding of a Ductile Metal under a General Stress State - Mises Yield Condition.

The work done per unit volume of material by the stress in a simple tension test is determined by the area under the stress-strain curve up to the level of strain experienced at max stress. In the elastic region (assuming linear behavior) this is given by

$$W = \frac{1}{2}\sigma\varepsilon$$

All of this work is stored as "elastic energy" in the material, which is released at unloading. Once the material is in the plastic region a part of this work is associated with changing the materials volume while the rest is associated with changing its shape (distorting).

This same division is also true in a general stress loading and forms the basis for the Mises Yield Condition expressing the condition for yielding, or elastic break-down, under general stress loading. The Mises Condition can be interpreted as requiring that yielding occurs when the strain energy of distortion in a general stress state equals the strain energy of distortion at yielding in a simple tension test on the same material.

To develop the mathematical form of this expression, we consider a general cubic element of material having principal stresses σ_1 , σ_2 , and σ_3 acting on it and experiencing associated strain ε_1 ,

ϵ_2 , and ϵ_3 . The work (or strain energy) per unit volume of material associated with these strains is given, by analogy with eq. (17) as

$$w = \frac{1}{2}\sigma_1\epsilon_1 + \frac{1}{2}\sigma_2\epsilon_2 + \frac{1}{2}\sigma_3\epsilon_3$$

Now, from the elastic stress strain relations we have

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E}(\sigma_2 + \sigma_3)$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_3)$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_2)$$

where E is Young's modulus, ν is Poisson's ratio, both constants for a given material. Substituting we find

$$w = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)$$

Now consider the volume of the small element of material. If its sides were initially of length a , its initial volume was a^3 and its final volume is $a^3(1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3)$. For small strains, we thus have for the change in unit volume per unit initial area

$$\Delta = (1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3) - 1 \approx \epsilon_1 + \epsilon_2 + \epsilon_3$$

We also see that

$$\Delta = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

Hence, the change in volume is related directly to the sum $\sigma_1 + \sigma_2 + \sigma_3$. We may then define the mean stress $\bar{\sigma}$ as

$$\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

The element stresses can be thought of as being composed of this mean stress $\bar{\sigma}$ and an additional stress dependent on the mean stress; i.e.

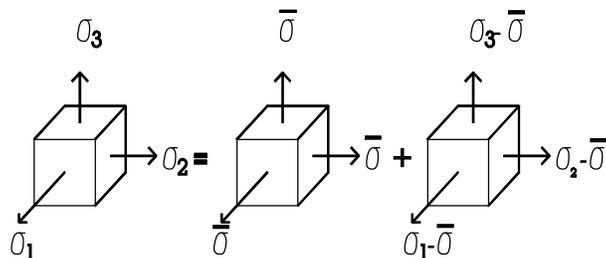


Figure 10

It is clear that the stress $\bar{\sigma}$ is associated with the total volume change of the material and the stresses $\sigma_1 - \bar{\sigma}$, $\sigma_2 - \bar{\sigma}$, $\sigma_3 - \bar{\sigma}$ are associated only with its distortion. Hence, the strain energy associated with the volume change is determined by substituting $\sigma_1 = \bar{\sigma}$, $\sigma_2 = \bar{\sigma}$, $\sigma_3 = \bar{\sigma}$ into the equation, we find

$$w_v = \frac{3}{2E}(1 - 2\nu)\bar{\sigma}^2$$

Subtracting this from the total strain energy w we find (using the definition of $\bar{\sigma}$) the strain energy of distortion given by

$$w - w_v = \frac{1 + \nu}{3E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3)$$

The value of this expression at yielding in a simple tension test ($\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_3 = 0$) is

$$w - w_v = \frac{1 + \nu}{3E}\sigma_Y^2$$

By equating the right-hand side of the previous equation to this value, we have the Mises Yield Condition in the form

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3 = \sigma_Y^2$$

An alternate form of this equation is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2$$

In two dimensional plane stress

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2$$

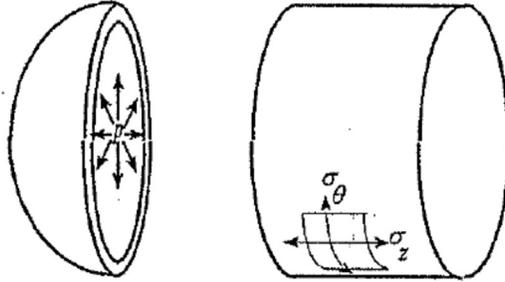
If we use for the two-dimensional case

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$$

we find the Mises Yield Condition expressible as

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = \sigma_Y^2$$

Example: A cylindrical pressure vessel of 15 inches in diameter has a wall thickness of 0.25 inches. The yield stress of the material is 40,000 psi. Determine the internal pressure required for yielding using the Mises Condition.



The stresses are

$$\sigma_\theta = (pr)/t, \sigma_z = (pr)/2t$$

where r = radius, t = wall thickness, p = net internal pressure. (these formulae are valid only for thin walled pressure vessels).

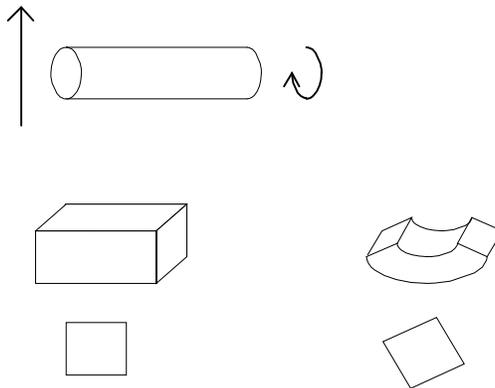
Solution: On substituting for r and t we have $\sigma_\theta = 30p$ and $\sigma_z = 15p$. These are principal stresses. Substituting into the Mises Condition, eq. (30), we have

$$900p^2 + 225p^2 - 450p^2 = \sigma_Y^2 = 40000 \text{ psi}$$

$$\text{or } p = 1540 \text{ psi}$$

This is the net internal pressure, i.e. the difference between internal and external pressure.

Example. A circular steel shaft is subjected to a bending moment of 3000 in-lb and a twisting moment of 8000 in-lb. Determine the required diameter of the shaft if the yield stress in a simple tension test is 36,000 psi.



Solution: the maximum bending stress σ_x is

$$\sigma_x = M_b r / I$$

where M_b is the bending moment and I is the moment of inertia of the cross-sectional area with respect to the neutral axis. Therefore

$$\sigma_x = (3000)(r) / (\pi r^4 / 4) = 3820 / r^3$$

The maximum shear stress is $\tau_{xy} = (M_t r) / (J)$ where M_t is the twisting moment and J is the polar moment of inertia. Therefore

$$\tau_{xy} = (8000)(r) / (\pi r^4 / 2) = 5093 / r^3$$

Substituting into the Mises relation

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = \sigma_Y^2$$

with $\sigma_y = 0$, we have

$$\left(\frac{3820}{r^3}\right)^2 + 3\left(\frac{5093}{r^3}\right)^2 = (36000)^2$$

Hence $r^6 = 0.0710 \text{ in}^6 \Rightarrow r = 0.644 \text{ in}$ and the required diameter is 1.29 in.

10.6 Maximum shear stress condition.

A theory for the yielding of ductile metals which is sometimes used in place of the Mises Yield Condition is the maximum shear stress condition. This theory assumes that elastic breakdown will occur when the maximum shear stress in a general stress state reaches the value it has in a simple tension test at yielding. Thus, if τ_{\max} denotes the maximum shear stress in a general stress loading, we have

$$\tau_{\max} = \frac{1}{2}\sigma_Y$$

where $\frac{1}{2}\sigma_Y$ is the maximum stress in a simple tension loading at yield.

Example: Reconsider the pressure vessel from the previous example using the maximum shear stress condition.

Solution: The principal stresses are $\sigma_1 = pr/t$, $\sigma_2 = pr/2t$, $\sigma_3 = 0$. The maximum shear stress is thus

$$\tau_{\max} = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{pr}{2t}$$

$$pr/2t = \frac{1}{2}\sigma_Y \text{ or } p = 1333 \text{ psi.}$$

Note that this is about 13% less than that given by the Mises equation and thus provides a more conservative estimate. Experimentally, the Mises Yield Condition has proven to be the most accurate.

10.7 Creep

“Creep” is a progressive plastic deformation that increases with time, even when the stress is below the yield stress of the material. The effects of creep increase as temperature increases, and generally noticeable at above 35% of the melting temperature of the material. Figure 11, below shows the effects of creep, in term of strain versus time.

- Initially, the rate of creep (slope) is high. In this initial stage, effects analogous to strain hardening reduce the rate of creep with time.
- In the second stage, the creep achieves a constant rate due to a balance between work hardening and annealing (thermal softening).
- In the third stage, necking of the material causes the strain to increase more rapidly, leading to rupture.

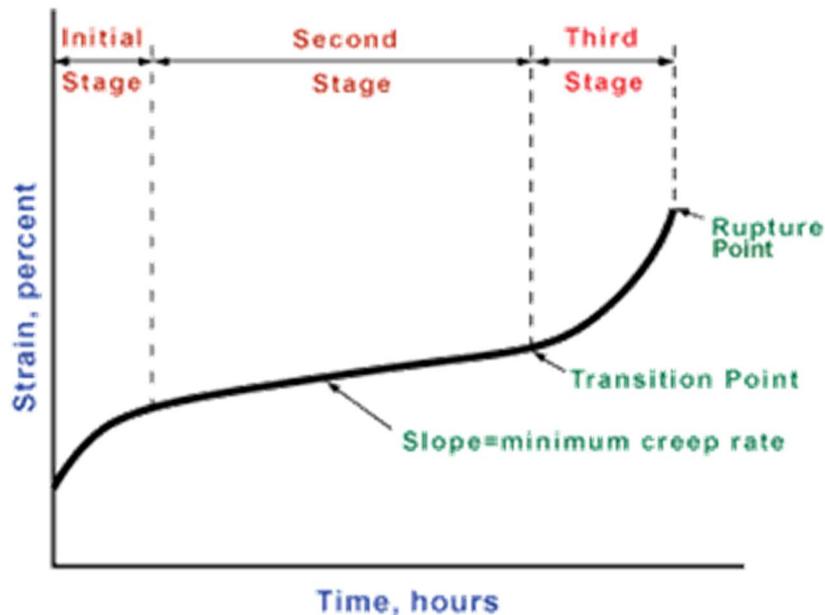


Figure 11: Stages of Creep

A well-known example of material creep was the World Trade Center Collapse. The stress on the materials increased because of the reduced structural capacity of the damaged building and added weight due to planes. The burning jet fuel increased the temperature, causing thermal creep. This led to a reduced cross section of the material and eventual failure.

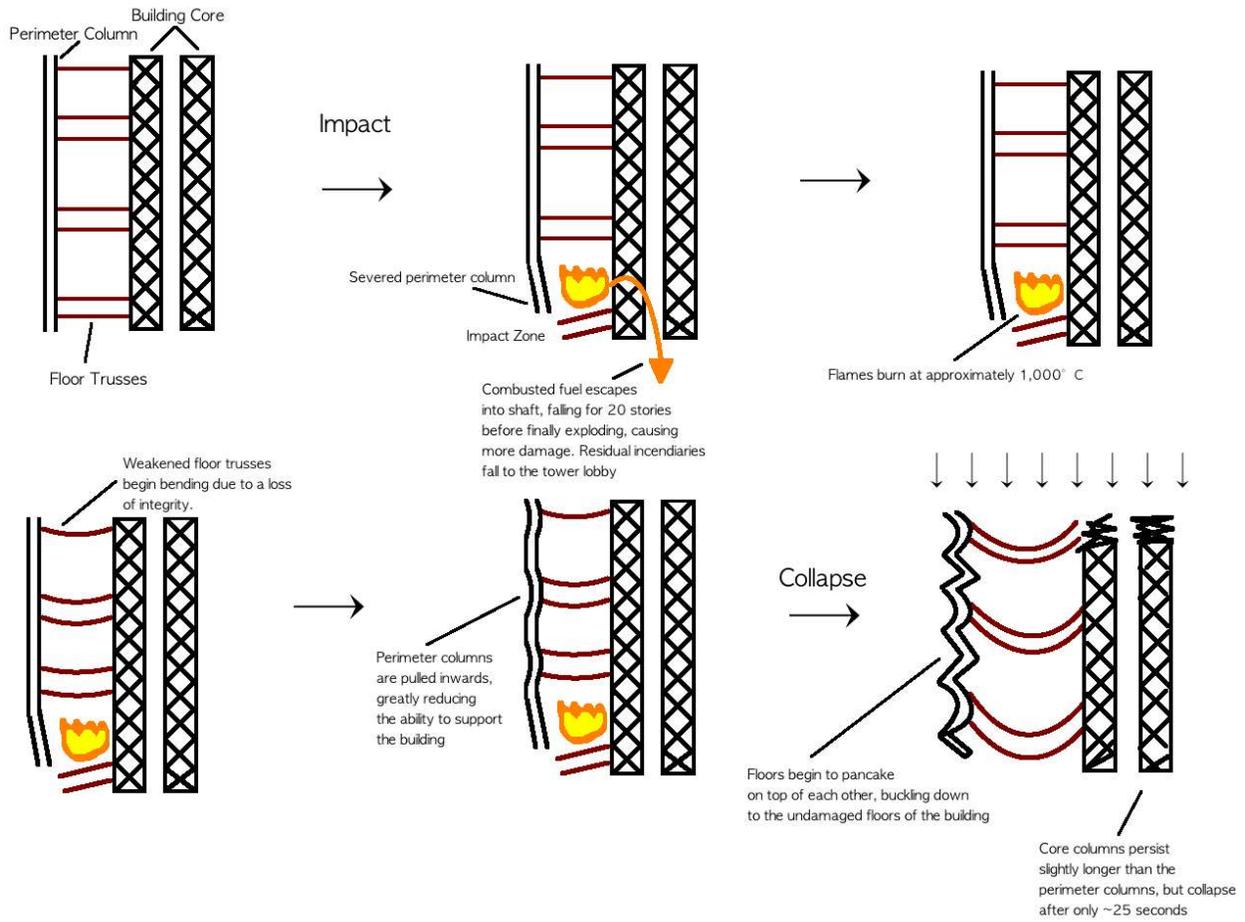


Figure 12: Creep and Failure mechanism in World Trade Center Collapse
(From TLC "Anatomy of Collapse")